Diversity-induced coherence resonance in spatially extended chaotic systems

Hanshuang Chen and Jiqian Zhang*
College of Physics and Electronic Information, Anhui Normal University, Wuhu, Anhui, 241000, China
(Received 13 September 2007; revised manuscript received 25 October 2007; published 11 February 2008)

The effect of parameter diversity on coupled Chua systems is investigated. In the absence of diversity, the systems jump back and forth between two variable domains of a chaotic attractor, and the residence times within a single domain are uncertain. By introducing parameter diversity, a combined numerical and analytical approach indicates that the systems can jump regularly from one domain to another at an intermediate range of diversity, a signature of coherence resonance. Furthermore, the influences of coupling strength and the number of units are also considered. Our results provide a possibility for the control of chaos in spatially extended chaotic systems by the manipulation of parameter diversity.

DOI: 10.1103/PhysRevE.77.026207 PACS number(s): 05.45.Gg, 05.40.Ca

I. INTRODUCTION

Noise-induced counterintuitive phenomena have attracted extensive attention over recent decades. Notably, stochastic resonance (SR) [1,2] and coherence resonance (CR) [2–4] have been widely investigated in experimental and theoretical work. SR is well known for the magnification of external forcing acting upon a nonlinear system under the presence of the right amount of noise, and CR refers to a resonantlike phenomenon of coherent motion that can be induced by noise only. Subsequently, more works have paid attention to the effect of noise on spatially extended systems [5]. Many interesting phenomena have been demonstrated, such as the enhancement of spatiotemporal coherence behavior of nonlinear systems by the optimization of a single adjustable parameter (noise or coupling) in an array of coupled systems; this effect is well known as array-enhanced stochastic resonance [6,7] and array-enhanced coherence resonance [8,9].

SR and CR have been studied experimentally and theoretically in chaotic systems whose dynamical trajectories have different preferred regions in phase space called chaotic attractors [10–15]. Thus, the chaotic system can be considered as a generalized bistable system, in contrast to the case of classical bistable systems. Even if additive or parametric noise is absent, SR has also been observed in the Lorenz model [16] and in the Rössler oscillator [17] as a result of chaos acting as an internal noise source.

In contrast to noise, diversity involves the fact that coupled units are not identical. This issue usually implies that the units composing the ensemble present a disparity in the values of some characteristic parameters, and has attracted much attention [18–28]. For instance, Braiman *et al.* have shown that diversity can enhance synchronization in an array of Josephson junctions [18]. Mousseau has demonstrated that the introduction of disorder can synchronize an integrate-and-fire coupled-map earthquake model [19]. Lindner *et al.* and Hou *et al.* have shown that an optimal magnitude of disorder, induced by the disparity of the pendulum lengths in an array of coupled pendulums, can order spatiotemporal chaos [20,21]. Recently, Tessone *et al.* reported that a disparity of the system's parameters could induce a resonant collective behavior in an ensemble of coupled bistable or excitable systems (*diversity-induced resonance*) [23]. Gassel *et al.* considered coupled forced FitzHugh-Nagumo oscillators, and achieved the optimal enhancement of the signal by both additive and multiplicative diversities (*double- diversity-induced resonance*) [27].

An intriguing problem is whether diversity can induce a resonant response in coupled chaotic systems. Therefore, for the purpose of the present work, the influence of parameter diversity on coupled Chua systems is discussed. We show numerically and analytically that parameter diversity can induce a resonant collective behavior, characterizing the periodic movement between two domains of a chaotic attractor in the presence of an intermediate level of diversity. These results represent the main feature of CR; thereby we summarize the diversity-induced CR in spatially extended chaotic systems.

II. MODEL DESCRIPTION

In the present work, we consider N coupled Chua systems whose dynamical equations are [29]

$$\dot{x}_i = \gamma_i[y_i - h(x_i)] + \frac{g}{N}\sum_{j=1}^{N}(x_j - x_i),$$

$$\dot{y}_i = x_i - y_i + z_i + \frac{g}{N}\sum_{j=1}^{N}(y_j - y_i),$$

$$\dot{z}_i = -\beta y_i + \frac{g}{N}\sum_{j=1}^{N}(z_j - z_i), \quad i = 1,2,\ldots,N, \tag{1}$$

where g denotes the coupling strength, and the nonlinear function $h(x)$ is given by $h(x) = bx + [(a-b)/2](|x+1| - |x-1|)$. Some of the parameters are taken as $\beta = 100/7$, $a = -1/7$, and $b = 2/7$. In order to introduce parameter diversity, we assume that the parameter γ_i is randomly selected from a uniform distribution $[\gamma_0 - \delta, \; \gamma_0 + \delta]$, where δ is the measurement of diversity. For an uncoupled Chua system with $\gamma = \gamma_0 = 9.0$, the system shows chaotic dynamics and the

*zhangcdc@mail.ahnu.edu.cn

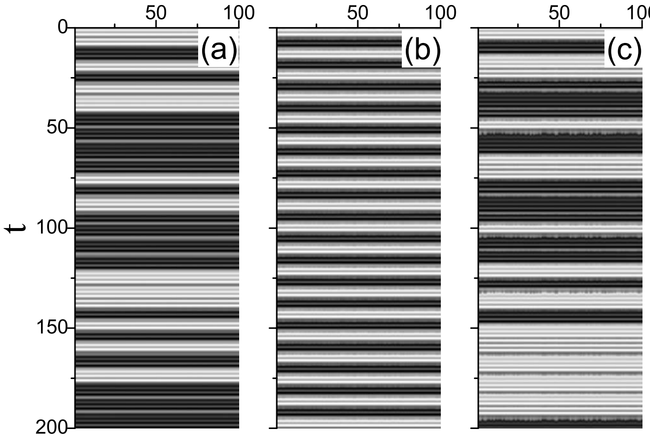

FIG. 1. Spatiotemporal evolution of x_i in $N=100$ coupled Chua systems with coupling strength $g=1.0$. The horizontal and vertical axes denote space and time, respectively. $\delta=$ (a) 0, (b) 0.08, and (c) 0.15.

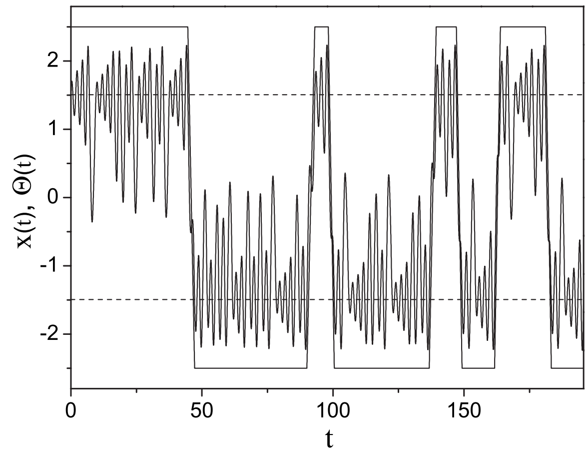

FIG. 2. Temporal evolution of variable x and its representation by the step function $\Theta(t)$ in an uncoupled Chua system. The dashed line indicates the crossing levels.

attractor has two variable domains (double-scroll structure). A single unit remains in each of these two scrolls for some indefinite time, and jumps chaotically between them. Numerical integration of Eq. (1) is performed by a fourth-order Runge-Kutta method with time steps of $dt=0.001$. To obtain each numerical result, we perform 50 runs that have different initial conditions. In each run, the first 10^6 time steps are discarded and 10^6 time steps are used to investigate the dynamics of the systems.

III. RESULTS AND DISCUSSION

The spatiotemporal evolution of the variable x_i in $N=100$ coupled Chua systems is displayed in Fig. 1. The x_i values are represented in gray scale; white corresponds to maximal values, black to minimal ones. In the absence of diversity ($\delta=0$), the systems jump between two scrolls and these jumps are irregular [see Fig. 1(a)]. If δ is increased to $\delta=0.08$, the regularity of jumps is obviously enhanced, as seen in Fig. 1(b), and the systems exhibit a rather periodic movement between the two scrolls. As δ is further increased to $\delta=0.15$, the regularity of jumps is destroyed again [see Fig. 1(c)].

Following the approach usually adopted in the literature [1,14,15], we replace the evolution of $x_i(t)$ by a two-state dynamics $\Theta(t)$, in which the detailed motion within each scroll is neglected: $\Theta(t)=\pm 2.5$ depending on the selected scroll. Usually, additional crossing levels, which in our case are at $x_i^C=\pm 1.5$, are used in this procedure. In Fig. 2, we exhibit the time evolution of the variable x and the step function $\Theta(t)$ in an uncoupled Chua system with $\gamma=\gamma_0=9.0$. The dashed line indicates the crossing levels.

According to the step function, we define a quantity to measure the regularity of the residence time T, i.e., the ratio of the averaged value to the standard deviation of the residence time $R=\langle T\rangle/\sqrt{\mathrm{var}(T)}$ [4,7,8,15,30]. A bigger R implies more regular jumps between two scrolls. The dependence of R on the diversity δ is depicted in Fig. 3. With an increment of δ, R reaches a maximum and decreases as δ increases further. Thus, there exists an optimal $\delta=\delta_{\mathrm{opt}}$ for the maximal $R=R_{\mathrm{max}}$, indicating the occurrence of diversity-induced CR in coupled chaotic systems.

Now, we consider the effects of coupling strength g and the number of units N on diversity-induced CR in coupled Chua systems. In Fig. 4, the dependence of R on δ for different g is exhibited. One can notice that the optimal δ_{opt} increases with an increment of g. However, the coupling strength must exceed a certain value to obtain this resonancelike behavior. By calculating the case of coupling strength $g=0.7$ and some smaller values, we find that this resonancelike behavior is absent (not shown in Fig. 4). In Fig. 5, the dependence of R on δ for different N is exhibited. One can observe that the location of the resonant peak does not depend on the number of units. On the other hand, the curve becomes smoother as N is increased, which is easy to recognize by comparing the curve for $N=100$ with the other three curves. For $N<100$, we perform numerical calculations at intervals of 10 from $N=10$ to 90. For $N\leq 50$, we find the curves become rather less smooth, and the resonant peak is

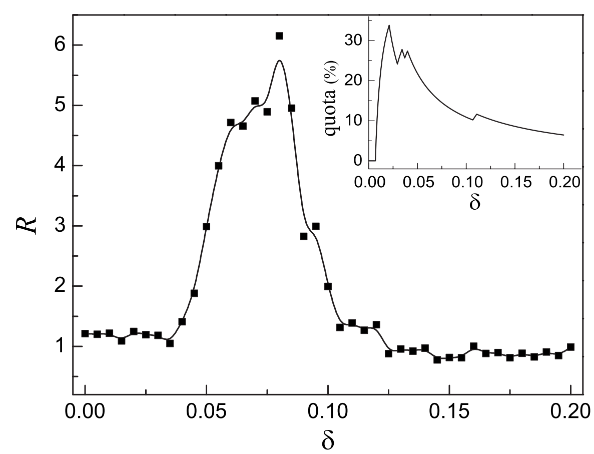

FIG. 3. Dependence of R on the diversity δ in $N=100$ coupled Chua systems with coupling strength $g=1.0$. The inset depicts the dependence of the quota of units in the periodic windows on the diversity δ.

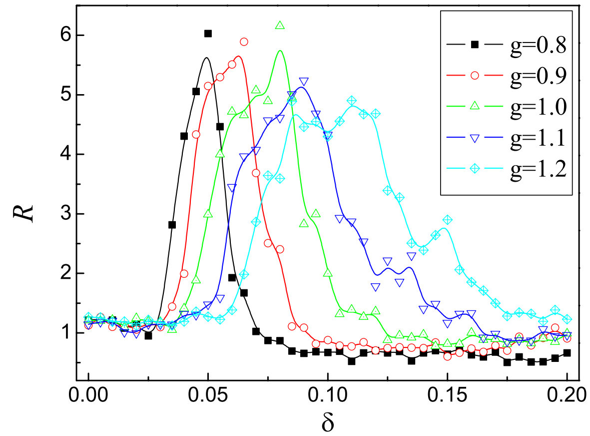

FIG. 4. (Color online) Dependence of R on the diversity δ for different coupling strengths g with $N=100$.

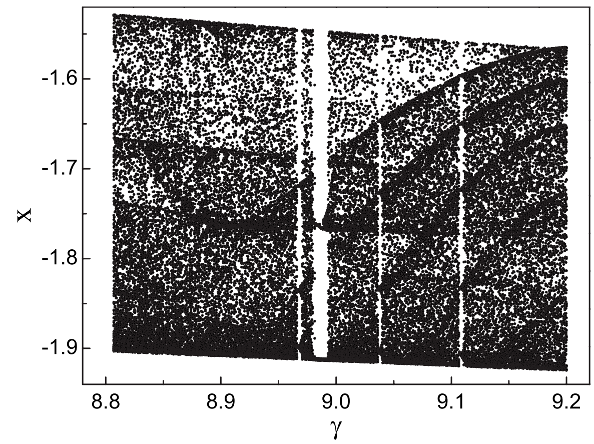

FIG. 6. Bifurcation diagram of an uncoupled Chua system. Some periodic windows can be observed.

hard to identify even if the numerical results are averaged over a greater number of runs. This is because for a quite small ensemble the parameter γ deviates badly from a uniform random distribution.

Let us consider a single Chua system. In Fig. 6, we plot the bifurcation diagram using the Poincaré section method, and in the present work we use the plane $z=1.5$. One can observe some periodic windows: [8.9658,8.9706] for period-3 orbits, [8.979,8.9932] for period-2 orbits, [9.037,9.0398] for period-2 orbits, and [9.1068,9.1108] for period-5 orbits. If the diversity δ is absent or very small, all units assembling the systems are located in chaotic states. Because a strong enough coupling interaction can synchronize all units, the systems behave as a chaotic oscillator whose jumps between two scrolls are irregular. As δ increases, more units that were originally located in the chaotic region move into periodic windows. If δ increases further, some units that were originally located in periodic windows enter the chaotic region again. Thus, an optimal δ should exist for the largest quota of units in the periodic windows. In the inset of Fig. 3, we draw the curve of the quota of units in the periodic windows as a function of the diversity δ. Comparison of the inset of Fig. 3 with Fig. 4 shows that the

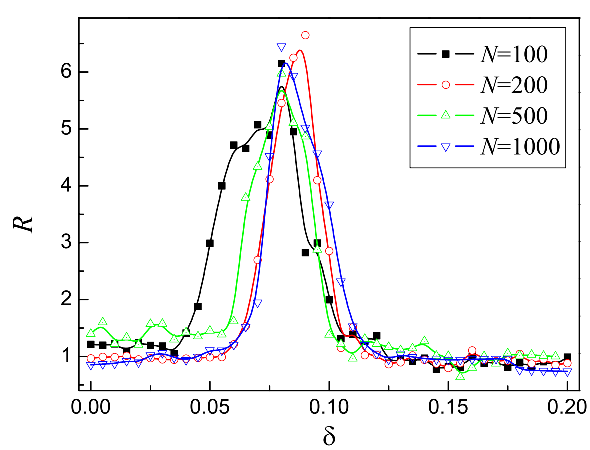

FIG. 5. (Color online) Dependence of R on the diversity δ for different numbers of units N with $g=1.0$.

location of the peak is distinctly different, especially for large values of the coupling strength. We can conclude that there is obviously no direct connection between the quota of units in the periodic windows and the observed resonance-like phenomenon, and the dynamics for the coexistence of periodic and chaotic oscillators is very complex.

To gain understanding of the above phenomena, we apply the order parameter expansion method to gain further understanding of our results. The order parameter expansion [31–33] is a useful approach for studying the changes in coherent behavior when the parameter mismatch is increased or the coupling strength is modified in populations of globally and strongly coupled elements. Let us briefly review the main results by considering a set of equations of the form

$$x_i = f(x_i,p_i) + K(X - x_i), \quad i = 1,2, \ldots ,N, \tag{2}$$

where p_i is a parameter different in every individual system. All the elements are coupled to the mean field of the population $X=\langle x_i\rangle$ through the coupling function $K(X-x_i)$. The basic idea of the method is to obtain an effective equation of motion for the mean-field variable X, valid when all the elements evolve in time close to X. Under this assumption, one gets the following approximate reduced equation for the mean field by averaging Eq. (2) over the population:

$$\dot{X} = F(X,p_0) + D_{x,p}F(X,p_0)W,$$

$$\dot{W} = \sigma^2 D_p F(X,p_0) + [D_x F(X,p_0) - K]W, \tag{3}$$

where $W=\langle (x_i-X)(p_i-p_0)\rangle$ is a second macroscopic variable or shape parameter, $p_0=\langle p_i\rangle$ is the average parameter value, and σ^2 is the variance of the parameter distribution. The derivative operators are defined as follows: $D_x=\partial/\partial x$, $D_p=\partial/\partial p$, and $D_{x,p}=\partial^2/\partial x\,\partial p$.

For the case of the present N globally coupled Chua equations [see Eq. (1)], one can easily introduce the reduced equations for the mean field $X=(X_x,X_y,X_z)^T$ and $W=(W_x,W_y,W_z)^T$, where $X_\alpha=\langle \alpha_i\rangle$, $W_\alpha=\langle(\alpha_i-X_\alpha)(\gamma_i-\gamma_0)\rangle$, $\alpha=x,y,z$, and $\gamma_0=\langle\gamma_i\rangle$:

$$\dot{X} = F(X,\gamma_0) + D_{x,\gamma}F(X,\gamma_0)W,$$

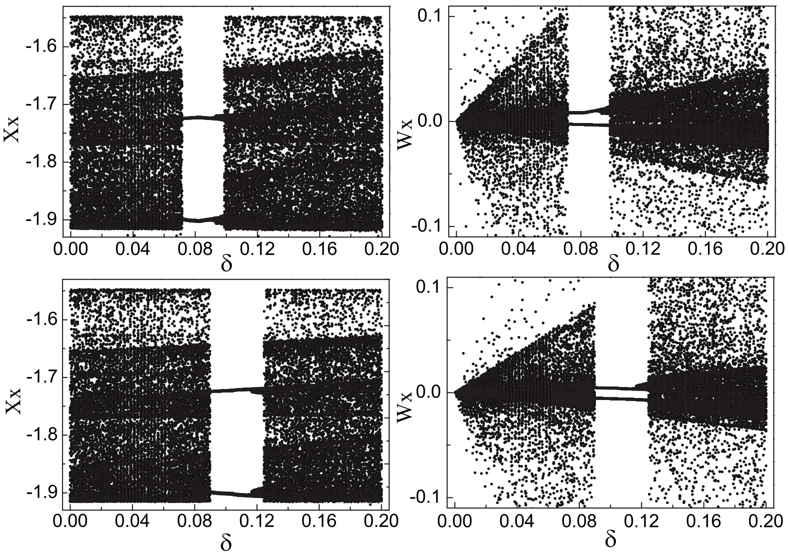

FIG. 7. Bifurcation diagram of the reduced system Eq. (4). Poincaré section of the mean field X_x and of the shape parameter W_x with coupling strength $g=1.0$ (top) and 1.2 (bottom).

$$\dot{W} = \sigma^2 D_\gamma F(X,\gamma_0) + [D_x F(X,\gamma_0) - K]W. \tag{4}$$

Here the vector $F(X,\gamma_0)=\{\gamma_0[X_y-h(X_x)],X_x-X_y+X_z,-\beta X_y\}^T$, and the Jacobian D_xF is

$$D_xF(X,\gamma_0)=\begin{pmatrix}\gamma_0 j_{11} & \gamma_0 & 0\\ 1 & -1 & 1\\ 0 & -\beta & 0\end{pmatrix},$$

where $j_{11}=-b-[(a-b)/2][\mathrm{sgn}(X_x+1)-\mathrm{sgn}(X_x-1)]$. $D_\gamma F$ and $D_{x,\gamma}F$ are

$$D_\gamma F(X,\gamma_0)=[X_y-h(X_x),0,0]^T,$$

$$D_{x,\gamma}F(X,\gamma_0)=\begin{pmatrix}j_{11} & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

The coupling matrix K is $K=\mathrm{diag}\{g,g,g\}$. Because the parameters γ_i are taken from a uniform distribution in the interval $[\gamma_0-\delta, \ \gamma_0+\delta]$, we obtain the mean value $\gamma_0=\langle\gamma_i\rangle=9.0$ and the variance $\sigma^2=\delta^2/3$, where δ denotes the diversity.

Using the Poincaré section method with the plane $z=1.5$, the bifurcation diagram of the reduced system Eq. (4) can be numerically calculated. We find that there exist coherent regimes in the bifurcation diagram when the coupling strength $g>0.7$, and the coherent regimes correspond with the resonant region in Fig. 4. For the case of $g\leq 0.7$, we do not find the existence of coherent regimes. For the case of $g>0.7$, the coherent regimes have the trend of a rightward shift as the coupling strength is increased, which confirms the numerical results of Fig. 4. Here we demonstrate the cases of coupling strength $g=1.0$ and 1.2 as seen in Fig. 7. The diagram provides coherent regimes around $\delta\approx 0.08$ for $g=1.0$ and $\delta\approx 0.11$ for $g=1.2$. One observes that all the coupled systems have period-2 oscillations in the coherent regime. These explanations agree well with numerical calculations, in particular, in the location of the resonant peak and its dependence on the coupling strength (Figs. 3 and 4). Since a small population size distorts the given parameter distribution, the application of the order parameter expansion method to the present work is valid only for large number of units, N. One can clearly discern in Fig. 5 that the numerical result for $N=1000$ fits the mean-field approximation better than for $N=100$. Therefore our analysis can be readily generalized to larger size.

It is useful to investigate the present used model when another parameter is selected for diversity and using similar dynamic equations, e.g., the well-known Lorenz equation. For the model Eq. (1), we consider the case where the parameter β is selected for diversity and the other parameters are fixed. The parameter β is assigned randomly from a uniform distribution along the units in the interval $[\beta_0-\delta, \ \beta_0+\delta]$, and the parameters are $\beta_0=100/7$, $\gamma=9.0$, $a=-1/7$, and $b=2/7$. For the coupled Lorenz equations $\dot{x}_i=p(y_i-x_i)+g(\langle x_i\rangle-x_i)$, $\dot{y}_i=-x_iz_i+r_ix_i-y_i$, and $\dot{z}_i=x_iy_i-bz_i+g(\langle z_i\rangle-z_i)$, the parameter r_i is assigned randomly from a uniform distribution along the units with the interval $[r_0-\delta, \ r_0+\delta]$, and the other parameters are $r_0=69.96$, $p=10$, $b=8/3$, and $g=6.0$. Similarly to the previous results, in Fig. 8, the numerical calculation shows that these systems also show a resonant response at the right diversity.

IV. CONCLUSION

In summary, we investigated numerically and analytically the effect of parameter diversity on coupled Chua systems. It is shown that the systems convert irregular jumps between two scrolls to regular jumps if an intermediate level of parameter diversity is applied. These results represent the main feature of CR. Therefore diversity-induced CR in coupled chaotic systems is exhibited in the present work. Furthermore, we demonstrated the effects of coupling strength and population size on diversity-induced CR. The location of the resonant peak is dependent on the coupling strength and independent of the population size. Importantly, only when the coupling strength exceeds a critical value is the resonance-like phenomenon present, and vice versa. We looked for an

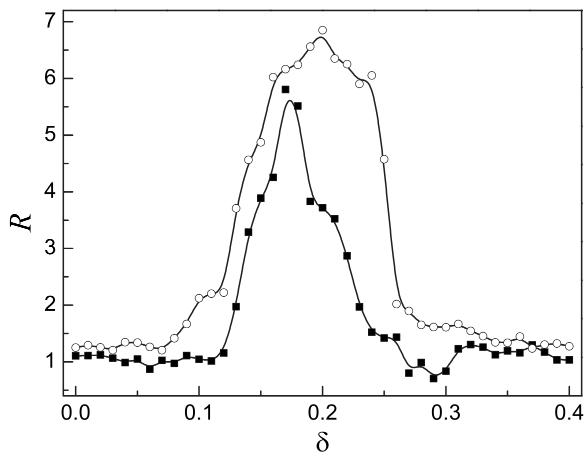

FIG. 8. Dependence of R on the diversity δ for Chua systems when the parameter β is selected for diversity with $g=1.0$ (solid squares), and for Lorenz systems with $g=6.0$ (hollow circles). The number of unit of the two systems is $N=100$.

analytical understanding by using the order parameter expansion method and obtain a good agreement with the numerical results. Diversity often influences our lives in many situations, like noise, so our results may have some interesting implications for the control of chaos in spatially extended chaotic systems by the manipulation of parameter diversity. We also expect experimental confirmation of the present work in the future.

ACKNOWLEDGMENTS

The authors thank the referees for their anonymous comments. This work was supported by the Educational Commission of Anhui Province of China (Grant No. KJ2007A079). The authors gratefully acknowledge the support of the Research Fund of Anhui Normal University (Grant No. 2006xzx09) and support by the Program for Innovative Research Teams in Anhui Normal University.

[1] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[2] B. Lindner, J. Garcia-Ojalvo, A. Neiman, and L. Schimansky-Geier, Phys. Rep. **392**, 321 (2004).
[3] Gang Hu, T. Ditzinger, C. Z. Ning, and H. Haken, Phys. Rev. Lett. **71**, 432 (1993).
[4] A. S. Pikovsky and J. Kurths, Phys. Rev. Lett. **78**, 775 (1997).
[5] F. Sagués, J. M. Sancho, and J. García-Ojalvo, Rev. Mod. Phys. **79**, 829 (2007).
[6] J. F. Lindner, B. K. Meadows, W. L. Ditto, M. E. Inchiosa, and A. R. Bulsara, Phys. Rev. Lett. **75**, 3 (1995).
[7] J. F. Lindner, B. K. Meadows, W. L. Ditto, M. E. Inchiosa, and A. R. Bulsara, Phys. Rev. E **53**, 2081 (1996).
[8] B. Hu and C. Zhou, Phys. Rev. E **61**, R1001 (2000).
[9] C. Zhou, J. Kurths, and B. Hu, Phys. Rev. Lett. **87**, 098101 (2001).
[10] V. S. Anishchenko, M. A. Safonova, and L. O. Chua, Int. J. Bifurcation Chaos Appl. Sci. Eng. **2**, 397 (1992).
[11] V. S. Anishchenko, M. A. Safonova, and L. O. Chua, Int. J. Bifurcation Chaos Appl. Sci. Eng. **4**, 441 (1994).
[12] G. Nicolis, C. Nicolis, and D. McKernan, J. Stat. Phys. **70**, 125 (1993).
[13] V. S. Anishchenko, A. B. Neiman, and M. A. Safanova, J. Stat. Phys. **70**, 183 (1993).
[14] W. Korneta, I. Gomes, C. R. Mirasso, and R. Toral, Physica D **219**, 93 (2006).
[15] C. Palenzuela, R. Toral, C. R. Mirasso, O. Calvo, and J. D. Gunton, Europhys. Lett. **56**, 347 (2001).
[16] A. Crisanti, M. Falcioni, G. Paladin, and A. Vulpiani, J. Phys. A **27**, L597 (1994).
[17] K. Arai, S. Mizutani, and K. Yoshimura, Phys. Rev. E **69**, 026203 (2004).
[18] Y. Braiman, W. L. Ditto, K. Wiesenfeld, and M. L. Spano, Phys. Lett. A **206**, 54 (1995).
[19] N. Mousseau, Phys. Rev. Lett. **77**, 968 (1996).
[20] Y. Braiman, J. F. Lindner, and W. L. Ditto, Nature (London) **378**, 465 (1995).
[21] J. F. Lindner, B. S. Prusha, and K. E. Clay, Phys. Lett. A **231**, 164 (1997).
[22] F. Qi, Z. Hou, and H. Xin, Phys. Lett. A **308**, 405 (2003).
[23] C. J. Tessone, C. R. Mirasso, R. Toral, and J. D. Gunton, Phys. Rev. Lett. **97**, 194101 (2006).
[24] C. J. Tessone, A. Scire, R. Toral, and P. Colet, Phys. Rev. E **75**, 016203 (2007).
[25] E. Glatt, M. Gassel, and F. Kaiser, Phys. Rev. E **75**, 026206 (2007).
[26] H. Chen, J. Zhang, and J. Liu, Phys. Rev. E **75**, 041910 (2007); Physica A **387**, 1071 (2008).
[27] M. Gassel, E. Glatt, and F. Kaiser, Phys. Rev. E **76**, 016203 (2007).
[28] G. H. Paissan and D. H. Zanette, Europhys. Lett. **77**, 20001 (2007).
[29] R. N. Madan, *Chua's Circuit: A Paradigm for Chaos* (World Scientific, Singapore, 1993).
[30] Y. Shinohara, T. Kanamaru, H. Suzuki, T. Horita, and K. Aihara, Phys. Rev. E **65**, 051906 (2002).
[31] S. D. Monte and F. d'Ovidio, Europhys. Lett. **58**, 21 (2002).
[32] S. De Monte, F. d'Ovidio, and E. Mosekilde, Phys. Rev. Lett. **90**, 054102 (2003).
[33] I. Gomes Da Silva, S. De Monte, F. d'Ovidio, R. Toral, and C. R. Mirasso, Phys. Rev. E **73**, 036203 (2006).